arXiv:gr-qc/0310093v1 19 Oct 2003

Surface terms, angular momentum and Hamilton - Jacobi formalism

Yurdahan Güler[1], **Dumitru Baleanu**[2], **Murat Cenk**[3]

Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankaya University-06530, Ankara , Turkey

Abstract

Quadratic Lagrangians are introduced adding surface terms to a free particle Lagrangian. Geodesic equations are used in the context of the Hamilton-Jacobi formulation of constrained sysytem. Manifold structure induced by the quadratic Lagrangian is investigated.

1 Introduction

There are mainly two basic methods to investigate constrained systems. First one is the method initiated by Dirac [1]. This method is based upon the classification of constraints and consistency conditions derived from the the variations of the constraints. The symplectic structure is established by defining Dirac brackets. The second method is the treatment of the constrained systems by Caratheodory's equivalent Lagrangian method [2],[3], [4]. Since this method leads us to the Hamilton-Jacobi equations it will be called Hamilton-Jacobi(HJ) approach. Equivalent Lagrangian method [2] leads us to a set of Hamilton-Jacobi equations [3], [4]. Despite of many attempts to clarify the many aspects of Hamilton-Jacobi formalism (HJ) we still have some subtle ones. The main difference between Dirac's formulation [1] and (HJ) approach [5], [6], [7] is the fact that the second one is multi Hamiltonian approach depending on the rank of the Hessian matrix. On the other hand the boundary conditions [8],[9], [10], [11] play an important role for constrained systems [12] and they are crucial for (HJ) formulation [2],[13]. The presence of many Hamiltonians, which are constraints having a special forms, suggests us to investigate the possibility to describe the superintegrable systems [14] and to obtain some integrable geometries.

In this paper we start with a free Lagrangian theory and we add a surface term involving the components of the angular momentum. By a suitable choosing of a Lagrange multipliers we obtain a quadratic Lagrangian which can be singular or not. The aim of this paper is to study quadratic, singular Lagrangians, using the (HJ) formulation. This paper is arranged as follows: In Sec. 2 (HJ) formalism

[1]E-Mail: yurdahan@cankaya.edu.tr

[2]On leave of absence from Institute of Space Sciences, P.O BOX, MG-23, R 76900 Magurele-Bucharest, Romania, E-mail: dumitru@cankaya.edu.tr

[3]E-Mail: mcenk@cankaya.edu.tr

is briefly explained. In Sec. 3 the method of obtaining quadratic Lagrangians is presented and two non-singular quadratic systems are studied. In Sec. 4 the singular case is analyzed using the (HJ) formalism. Conclusions are presented in Sec. 5. In Annex some geometrical properties of the obtained metrics are shown.

2 Hamilton-Jacobi formalism

Let us assume that the Lagrangian L is singular and the Hessian matrix $\frac{\partial^2 L}{\partial \dot{q^a} \partial \dot{q^b}}$, $a, b = 1, 2, .., n-r$, has rank n-r.The canonical Hamiltonian H_0 is defined as

$$H_0 = p_i \dot{q^i} - L(t, q^i, \dot{q^i})$$

The other Hamiltonians are obtained from the definitions $p_\alpha = \frac{\partial L}{\partial \dot{q^\alpha}} \mid_{\dot{q}_a = \omega_a}$ and they have the forms

$$H^{'}_\alpha = H_\alpha(t_\beta, q_a, p_a) + p_\alpha \tag{1}$$

where $\alpha, \beta = n-r+1, .., n; \quad a = 1, 2, .., n-r$. Thus, one arrives at the following set of Hamilton-Jacobi partial differential equations

$$\frac{\partial S}{\partial q^\alpha} + H_\alpha(t^\beta, q^a, p_a = \frac{\partial S}{\partial q^a}) = 0 \tag{2}$$

The following total differential equations lead us to determine the Hamilton-Jacobi function $S(t^\alpha, q^a)$:

$$dq^a = \frac{\partial H^{'}_\alpha}{\partial p_a} dt^\alpha \tag{3}$$

$$dp_a = -\frac{\partial H^{'}_\alpha}{\partial q^a} dt^\alpha \tag{4}$$

$$dp_\mu = -\frac{\partial H^{'}_\alpha}{\partial t^\mu} dt^\alpha \tag{5}$$

Solutions of these equations $S(q_i)$ as

$$dz = (-H_\alpha + p_a \frac{\partial H^{'}_\alpha}{\partial p_a}) dt^\alpha, \tag{6}$$

where $z = S(t^\alpha, q^a)$. Since the equations of motion are total differential equations, one should check the differentiability conditions, consistency conditions in Dirac formulation. It is proved that the system of equations (3) is differentiable if and only if the following Poisson brackets equations are satisfied :

$$[H^{'}_\alpha, H^{'}_\beta] = 0. \tag{7}$$

In general some of these Poisson brackets may not vanish identically. In such cases one should include the new functions as "new" constraints. Poisson brackets of these constraints with $H^{'}_\alpha$ should be considered also. This procedure should continue until all Poisson brackets vanish identically. This approach was studied in detail in references [3],[4].

3 The method

Let us assume that a given Lagrangian $L(\dot{q}^i, q^i) = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$ admits a set of independent constants of motion denoted by $L_i, i = 1, \cdots n$. Our aim is to treat the new Lagrangian $L^{'} = L + \dot{\lambda}^i L_i$ as a quadratic Lagrangian expresses as

$$L^{'} = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j. \tag{8}$$

Let us assume that the matrix g_{ij} is non-singular.

Treating g_{ij} as the "metric tensor" of a manifold the geodesic equations

$$\ddot{q}^a = \frac{-1}{2}g^{ab}(\frac{\partial g_{bm}}{\partial q^k} + \frac{\partial g_{bk}}{\partial q^m} - \frac{\partial g_{mk}}{\partial q^b})\dot{q}^k\dot{q}^m \tag{9}$$

are equivalent to the Euler-Lagrange equations of (8). Here g^{ab} is the inverse of g_{ab}. In this formulation the canonical Hamiltonian of the system is expressed as $H = \frac{1}{2}g^{ab}p_a p_b$. In this way the motion of a particle is described by geodesic equations of a Riemannian manifold with metric tensor g_{ij}. The above motion is treated inside of (HJ) formalism and the geometrical properties of the obtained metrics g_{ij} will be investigated. In the following we will add to a free Lagrangian the components of the angular momentum.

1.As the first example of a non-singular quadratic system, let's consider the following Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \dot{\lambda}_3(x\dot{y} - y\dot{x}), \tag{10}$$

which can be expressed as $L = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$, where

$$a_{ij} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{pmatrix}. \tag{11}$$

This Lagrangian corresponds to the motion of a particle moving on a plane such that L_z is constant.
Hamiltonian has the following form

$$H_c = \frac{1}{2}a^{-1}_{\mu\rho}p_\mu p_\rho \quad where \quad \mu, \rho = 1, .., 3 \tag{12}$$

and $a^{-1}_{\mu\rho}$ is the inverse of $a_{\mu\rho}$. From Hamiltonian we find

$$\begin{gathered} dx = a^{-1}_{1\rho}p_\rho dt, \quad dy = a^{-1}_{2\rho}p_\rho dt, \quad d\lambda_3 = a^{-1}_{3\rho}dt \\ dp_x = \tfrac{-1}{2}(\tfrac{\partial a^{-1}_{\mu\rho}}{\partial x})p_\mu p_\rho dt, \quad dp_y = \tfrac{-1}{2}(\tfrac{\partial a^{-1}_{\mu\rho}}{\partial y})p_\mu p_\rho dt, \quad dp_{\lambda_3} = 0 \end{gathered} \tag{13}$$

In order to find equation of motion we must solve (13). However we will prefer geodesic equations that produce easier equations and equivalent to the (13). The geodesic equations corresponding to (11) are

$$\begin{aligned} \ddot{x} &= \tfrac{2x\dot{\lambda}_3}{x^2+y^2}(x\dot{y}-y\dot{x}), \\ \ddot{y} &= \tfrac{2y\dot{\lambda}_3}{x^2+y^2}(x\dot{y}-y\dot{x}), \\ \ddot{\lambda}_3 &= \tfrac{-2\dot{\lambda}_3}{x^2+y^2}(x\dot{x}+y\dot{y}). \end{aligned} \tag{14}$$

The aim is to solve the system of equations given by (14). Notice that (14) implies that

$$\frac{\ddot{\lambda}_3}{\dot{\lambda}_3} = \frac{-2(x\dot{x}+y\dot{y})}{x^2+y^2}, \tag{15}$$

which admits solutions as

$$\lambda_3 = \int \frac{C}{x^2+y^2}dt, \tag{16}$$

where C is constant. Taking into account that

$$\frac{\ddot{x}}{\ddot{y}} = \frac{x}{y} \quad and \quad denoting \quad \frac{\ddot{x}}{x} = \frac{\ddot{y}}{y} = k$$

we divide our problem into two parts. The first one corresponds to the case when $k < 0$. We obtain the solutions as

$$x(t) = c_1\sin(\sqrt{k}t) + c_2\cos(\sqrt{k}t), \quad y(t) = c_3\sin(\sqrt{k}t) + c_4\cos(\sqrt{k}t). \tag{17}$$

If we impose some initial conditions one can find specific solutions of equations.Let us assume that $c_1 = 0$, $c_2 = 1$, $c_3 = 1$ and $c_4 = 0$. From (17) we find that $x(t) = \cos(\omega t)$ *and* $y(t) = \sin(\omega t)$ where $\omega = \sqrt{k}$. Also from equation (16) we obtain that $\lambda_3 = Ct + B$ where B is a constant. On the other hand p_x, p_y and p_{λ_3} are given as follows

$$\begin{pmatrix} p_x \\ p_y \\ p_{\lambda_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\lambda}_3 \end{pmatrix} \tag{18}$$

If we solve (18) we find

$$p_x = \sin(\omega t)(-\omega - C) \quad p_y = \cos(\omega t)(\omega + C) \quad p_{\lambda_3} = \omega \tag{19}$$

In the case of $k > 0$ the solutions are given by

$$x(t) = c_1e^{\sqrt{k}t} + c_2e^{-\sqrt{k}t}, y(t) = c_3e^{\sqrt{k}t} + c_4e^{-\sqrt{k}t}, \tag{20}$$

where $c_i, i = 1, .., 4$ are constants. As in the previous case λ_3 is given by (16).Using (20) and imposing $A = c_3c_2 - c_1c_4 \neq 0$ we obtain the following equation $(c_4^2 + c_3^2)x^2 + (c_1^2 + c_2^2)y^2 - 2xy(c_2c_4 + c_1c_3) = A^2$.

The metric from (11) is conformaly flat having its Ricci scalar $R = \frac{2}{x^2+y^2}$.

2. Let us add now two components of the angular momentum to the Lagrangian of a free three-dimensional particle. We obtain

$$L = \frac{1}{2}(\dot{x^2} + \dot{y^2} + \dot{z^2}) + \dot{\lambda_1}(y\dot{z} - z\dot{y}) + \dot{\lambda_2}(z\dot{x} - x\dot{z}), \tag{21}$$

and from (21) we identify the metric a_{ij} as

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & -z & 0 \\ 0 & 0 & 1 & y & -x \\ 0 & -z & y & 0 & 0 \\ z & 0 & -x & 0 & 0 \end{pmatrix} \tag{22}$$

In this case the geodesic equations have the following forms

$$\ddot{x} = \frac{x}{x^2+y^2+z^2}[2\dot{\lambda_1}(y\dot{z} - z\dot{y}) + 2\dot{\lambda_2}(z\dot{x} - x\dot{z})] \tag{23}$$

$$\ddot{y} = \frac{y}{x^2+y^2+z^2}[2\dot{\lambda_1}(y\dot{z} - z\dot{y}) + 2\dot{\lambda_2}(z\dot{x} - x\dot{z})] \tag{24}$$

$$\ddot{z} = \frac{z}{x^2+y^2+z^2}[2\dot{\lambda_1}(y\dot{z} - z\dot{y}) + 2\dot{\lambda_2}(z\dot{x} - x\dot{z})] \tag{25}$$

$$\ddot{\lambda_1} = \frac{2}{x^2+y^2+z^2}[-\dot{\lambda_1}(\frac{x^2\dot{z} + z^2\dot{z}}{z} + y\dot{y}) + \dot{\lambda_2}y(\frac{z\dot{x} - x\dot{z}}{z})] \tag{26}$$

$$\ddot{\lambda_2} = \frac{2}{x^2+y^2+z^2}[x\dot{\lambda_1}(\frac{z\dot{y} - y\dot{z}}{z} +) - \dot{\lambda_2}(\frac{y^2\dot{z} + z^2\dot{z}}{z} + x\dot{x})] \tag{27}$$

These equations form a system of nonlinear ordinary differential equation and to solve this system we should notice that $\frac{\ddot{x}}{x} = \frac{\ddot{y}}{y} = \frac{\ddot{z}}{z}$. Assuming that the above ratio is a negative constant $k^{'}$ we obtain

$$x(t) = c_1 \sin(\sqrt{k}t) + c_2 \cos(\sqrt{k}t) \tag{28}$$

$$y(t) = c_3 \sin(\sqrt{k}t) + c_4 \cos(\sqrt{k}t) \tag{29}$$

$$z(t) = c_5 \sin(\sqrt{k}t) + c_6 \cos(\sqrt{k}t), \tag{30}$$

where $-k^{'} = k$ and the constants $c_i, \quad i = 1...4$ are subjected to the restriction that $\frac{2\dot{\lambda_1}(y\dot{z}-z\dot{y})+2\dot{\lambda_2}(z\dot{x}-x\dot{z})}{x^2+y^2+z^2}$ is a constant. Note that if $k = 0$, both $\ddot{x}$ and $\ddot{y}$ must be 0, then x and y are both linear function of t. λ_1 and λ_2 can be easily obtained from (26) and (27).For the general case the expressions involved are very complicated. Instead of doing this we will present a particular solution. If we assume that $c_2 = \frac{\sqrt{2}}{2}$, $c_3 = 1$, $c_6 = \frac{\sqrt{2}}{2}$ and others are 0 , we obtain

$$x(t) = \frac{\sqrt{2}}{2}\cos(\sqrt{k}t), \quad y(t) = sin(\sqrt{k}t) \quad and \quad z(t) = \frac{\sqrt{2}}{2}\cos(\sqrt{k}t). \tag{31}$$

Using (31), (26) and (27) we obtain

$$
\begin{aligned}
\ddot{\lambda}_1(t) &= 0 && (32)\\
\ddot{\lambda}_1(t) &= \sqrt{2k}\dot{\lambda}_1(t) - \frac{2sin(\sqrt{k}t)\sqrt{k}}{\cos(\sqrt{k}t)}\dot{\lambda}_2(t). && (33)
\end{aligned}
$$

The solution of (32) is

$$
\begin{gathered}
\lambda_1(t) = k_3 t + k_4,\\
\lambda_2(t) = \tfrac{1}{4}k_3 ln(1 + tan^2(\sqrt{k}t))\sqrt{2}\sqrt{\tfrac{1}{k}} + \tfrac{1}{2}k_3 ln(\cos(\sqrt{k}t))\sqrt{2}\sqrt{\tfrac{1}{k}}\\
+\tfrac{1}{4\sqrt{k}}k_1 sin(2\sqrt{k}t) + (\tfrac{\sqrt{2}}{4\sqrt{k}} - \tfrac{\sqrt{2}}{4k}\cos(2\sqrt{k}t))k_3 + k_2 + \tfrac{1}{2}k_1 t
\end{gathered} \tag{34}
$$

where $k_i, i = 1, \cdots 4$ are constants. As before p_i have the forms $p_i = a_{ij}\dot{q^j}$.

4 The singular case

As an example of singular quadratic system, let's consider the Lagrangian given by

$$
L = \frac{1}{2}(\dot{x^2} + \dot{y^2} + \dot{z^2}) + \dot{\lambda_1}(y\dot{z} - z\dot{y}) + \dot{\lambda_2}(z\dot{x} - x\dot{z}) + \dot{\lambda_3}(x\dot{y} - y\dot{x}) \tag{35}
$$

which is expressed in compact form as $L = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$, where the matrix a_{ij} has the form

$$
a_{ij} = \begin{pmatrix}
1 & 0 & 0 & 0 & z & -y\\
0 & 1 & 0 & -z & 0 & x\\
0 & 0 & 1 & y & -x & 0\\
0 & -z & y & 0 & 0 & 0\\
z & 0 & -x & 0 & 0 & 0\\
-y & x & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{36}
$$

Since the rank of the matrix is 5, this is a singular system. By using $p_i = \frac{\partial L}{\partial \dot{q^i}}$ we find

$$
p_x + y\dot{\lambda_3} = \dot{x} + z\dot{\lambda_2}, \quad p_y - x\dot{\lambda_3} = \dot{y} - z\dot{\lambda_1}, p_z = a_{3i}\dot{q}^i, \quad p_{\lambda_1} = a_{4i}\dot{q}^i, \quad p_{\lambda_2} = a_{4i}\dot{q}^i. \tag{37}
$$

From (37) we find

$$
\begin{gathered}
\dot{x} = \frac{p_x zx^2 + p_{\lambda_2}z^2 + p_{\lambda_2}y^2 + xp_z z^2 + xyp_{\lambda_1} + xyzp_y}{z(x^2+y^2+z^2)},\\
\dot{y} = \frac{p_y zy^2 + p_z z^2 y + p_{\lambda_1}z^2 + xyzp_x - xyp_{\lambda_2} - x^2 p_{\lambda_1}}{z(x^2+y^2+z^2)},\\
\dot{z} = \frac{p_z z^2 + p_y yz + p_{\lambda_1}y - x\lambda_2 - xzp_x}{z(x^2+y^2+z^2)},\\
\dot{\lambda_1} = \frac{-p_y z^3 + z^3 x\dot{\lambda_3} + p_z z^2 y - p_{\lambda_1}z^2 + xyzp_x + x^3 z\dot{\lambda_3} - zx^2 p_y + zx\dot{\lambda_3}y^2 - xyp_{\lambda_2} - x^2 p_{\lambda_1}}{z(x^2+y^2+z^2)},\\
\dot{\lambda_2} = \frac{-p_{\lambda_2}z^2 - p_{\lambda_2}y^2 z^3 p_x + zy^2 p_x + z^3 y\dot{\lambda_3} + y^3 z\dot{\lambda_3} - p_z xz^2 - p_{\lambda_1}xy - p_y zyx + x^2 yz\dot{\lambda_3}}{z(x^2+y^2+z^2)}.
\end{gathered} \tag{38}
$$

Using (38), the canonical Hamiltonian has the following form

$$H_c = \frac{1}{2} b^{-1}_{\mu\rho} p_\mu p_\rho + \dot{\lambda}_3 (p_{\lambda_3} + \frac{y p_{\lambda_2} + x p_{\lambda_1}}{z}) \quad where \quad \mu, \rho = 1, .., 5 \tag{39}$$

where $b^{-1}_{\mu\rho}$ is the inverse of

$$b_{\mu\rho} = \begin{pmatrix} 1 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & -z & 0 \\ 0 & 0 & 1 & y & -x \\ 0 & -z & y & 0 & 0 \\ z & 0 & -x & 0 & 0 \end{pmatrix} \tag{40}$$

In (HJ) formalism the Hamiltonians to start with are

$$H'_0 = p_0 + \frac{1}{2} b^{-1}_{\mu\rho} p_\mu p_\rho, \quad H'_1 = p_{\lambda_3} + \frac{y p_{\lambda_2} + x p_{\lambda_1}}{z}. \tag{41}$$

The total differential equation corresponding to (41)are

$$dx = b^{-1}_{1\rho} p_\rho dt, \quad dy = b^{-1}_{2\rho} p_\rho dt, \quad dz = b^{-1}_{3\rho} p_\rho dt, \tag{42}$$

$$d\lambda_1 = b^{-1}_{4\rho} p_\rho dt + \frac{x}{z} d\lambda_3, \quad d\lambda_2 = b^{-1}_{5\rho} dt + \frac{y}{z} d\lambda_3 \tag{43}$$

$$dp_{\lambda_1} = 0, \quad dp_{\lambda_2} = 0, \quad dp_x = \frac{-1}{2} (\frac{\partial b^{-1}_{\mu\rho}}{\partial x}) p_\mu p_\rho dt - \frac{p_{\lambda_1}}{z} d\lambda_3, \tag{44}$$

$$dp_y = \frac{-1}{2} (\frac{\partial b^{-1}_{\mu\rho}}{\partial y}) p_\mu p_\rho dt - \frac{p\lambda_2}{z} d\lambda_3, \tag{45}$$

$$dp_z = \frac{-1}{2} (\frac{\partial b^{-1}_{\mu\rho}}{\partial z}) p_\mu p_\rho dt + (\frac{y p_{\lambda_2} + x p_{\lambda_2}}{z^2}) d\lambda_3 \tag{46}$$

The next step is to find the variations of H'_0 and H'_1. Using the fact that $b^{-1}_{\mu\rho}$ is a symmetric matrix and taking into account the previous equations we obtain after some calculations that both variations are zero. We conclude that the system is integrable and we observe that $H'_1 = z\vec{\mathbf{r}} x \vec{\mathbf{L}}$ which is identically zero. The geometrical properties of the $5X5$ metrics are presented in Annex.

4.1 Curved space generalization

Let us start with a free three dimensional Lagrangian

$$L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{47}$$

If we impose $z^2 = 1 - x^2 - y^2$ and take its time derivative, we obtain $\dot{z} = -\frac{x\dot{x} + y\dot{y}}{\sqrt{u}}$. So, (47) becomes

$$L' = g_{ab} \dot{q}^a \dot{q}^b, \tag{48}$$

where

$$g_{ab} = \delta_{ab} + \frac{q^a q^b}{u}. \tag{49}$$

where $z^2 = u, \quad a, b = 1, 2$. The above metric admits three invariants [15]

$$L_x = -\sqrt{u} p_y, L_y = \sqrt{u} p_x, L_z = x p_y - y p_x, \tag{50}$$

Adding those three constants of motion to (48) we obtain the extended Lagrangian in a compact form as $L^{'} = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j$, where a_{ij} is has the following form

$$a_{ij} = \begin{pmatrix} 1 + \frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}} + \sqrt{u} & -y \\ \frac{xy}{u} & 1 + \frac{y^2}{u} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & \frac{xy}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & 0 & 0 & 0 \\ \frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}. \tag{51}$$

The determinant a_{ij} is 0 so a_{ij} is singular symmetric matrix and the rank of it is 4. In order to find the canonical Hamiltonian we need $\dot{x}$, $\dot{y}$, $\dot{\lambda_1}$ and $\dot{\lambda_2}$ in terms of p_x, p_y, p_{λ_1}, p_{λ_2} and $\dot{\lambda_3}$ .Using

$$p_i = \frac{\partial L}{\partial \dot{q^i}} = a_{ij} \dot{q^j} \quad where \quad i, j = 1, .., 5, \tag{52}$$

we obtain

$$\begin{aligned} \dot{x} &= \frac{(-p_{\lambda_2} + p_{\lambda_2} x^2 + xy p_{\lambda_1}) \sqrt{1-x^2-y^2}}{1-x^2-y^2} \\ \dot{y} &= \frac{(p_{\lambda_1} - p_{\lambda_1} y^2 + xy p_{\lambda_2}) \sqrt{1-x^2-y^2}}{1-x^2-y^2} \\ \dot{\lambda_1} &= \frac{x^3 \dot{\lambda_3} + x^3 y p_x - x^2 p_y + x^2 p_y y^2 + x y^2 \dot{\lambda_3} - x \dot{\lambda_3} + xy \sqrt{1-x^2-y^2} + x y^3 p_x}{\sqrt{1-x^2-y^2}(1-x^2-y^2)} \\ &\quad - \frac{xy p_x - p_{\lambda_1} \sqrt{1-x^2-y^2} y^2 - 2 p_y y^2 + p_{\lambda_1} \sqrt{1-x^2-y^2} + y^4 p_y + p_y}{\sqrt{1-x^2-y^2}(1-x^2-y^2)} \\ \dot{\lambda_2} &= \frac{p_{\lambda_2} \sqrt{1-x^2-y^2}(x^2-1) - p_x + 2x^2 p_x - y \dot{\lambda_3} + x^2 y \dot{\lambda_3} - x^4 p_x + y^2 p_x - y^2 x^2 p_x}{\sqrt{1-x^2-y^2}(1-x^2-y^2)} \\ &\quad + \frac{y^3 \dot{\lambda_3} - x y^3 p_y + xy p_{\lambda_1} \sqrt{1-x^2-y^2-x^3 y p_y + xy p_y}}{\sqrt{1-x^2-y^2}(1-x^2-y^2)} \end{aligned} \tag{53}$$

so, the canonical Hamiltonian is given by

$$H_c = \frac{1}{2} g^{-1}_{\mu\rho} p_\mu p_\rho + \dot{\lambda_3}(p_{\lambda_3} + \frac{y p_{\lambda_2} + x p_{\lambda_1}}{\sqrt{1-x^2-y^2}}) \quad where \quad \mu, \rho = 1, .., 4 \tag{54}$$

where

$$g_{\mu\rho} = \begin{pmatrix} 1 + \frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}} + \sqrt{u} \\ \frac{xy}{u} & 1 + \frac{y^2}{u} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & \frac{xy}{\sqrt{u}} \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}} - \sqrt{u} & 0 & 0 \\ \frac{x^2}{\sqrt{u}} + \sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 \end{pmatrix}. \tag{55}$$

Since the matrix $g_{\mu\rho}$ is singular, we obtain a constrained system. In (HJ) formalism we have two Hamiltonians

$$H_0^{'} = p_0 + \frac{1}{2} g_{\mu\rho}^{-1} p_\mu p_\rho, \quad H_1^{'} = p_{\lambda_3} + \frac{y p_{\lambda_2} + x p_{\lambda_1}}{\sqrt{1 - x^2 - y^2}} \tag{56}$$

The total differential equations of (56) are given by

$$\begin{gathered} dx = g_{1\rho}^{-1} p_\rho dt, \quad dy = g_{2\rho}^{-1} p_\rho dt, \quad d\lambda_1 = g_{3\rho}^{-1} dt + \frac{x}{\sqrt{1-x^2-y^2}} d\lambda_3, \\ d\lambda_2 = g_{4\rho}^{-1} dt + \frac{y}{\sqrt{1-x^2-y^2}} d\lambda_3, \\ dp_x = \frac{-1}{2}\left(\frac{\partial g_{\mu\rho}^{-1}}{\partial x}\right) p_\mu p_\rho dt - \left(\frac{p\lambda_1}{\sqrt{1-x^2-y^2}} + \frac{(x p_{\lambda_1} + y p_{\lambda_2})x}{(1-x^2-y^2)^{\frac{3}{2}}}\right) d\lambda_3 \\ dp_y = \frac{-1}{2}\left(\frac{\partial g_{\mu\rho}^{-1}}{\partial y}\right) p_\mu p_\rho dt - \left(\frac{p\lambda_2}{\sqrt{1-x^2-y^2}} + \frac{(x p_{\lambda_1} + y p_{\lambda_2})y}{(1-x^2-y^2)^{\frac{3}{2}}}\right) d\lambda_3 \\ dp_{\lambda_1} = 0, dp_{\lambda_2} = 0 \end{gathered} \tag{57}$$

The variations of $H_0^{'}$ and $H_1^{'}$ are 0. Geodesic equations corresponding to $g_{\mu\rho}$ are given by

$$\ddot{y}u = y(y^2 - 1)\dot{x}^2 - 2xy^2\dot{x}\dot{y} + y(x^2 - 1)\dot{y}^2 \tag{58}$$

$$\ddot{x}u = x(y^2 - 1)\dot{x}^2 - 2yx^2\dot{x}\dot{y} + x(x^2 - 1)\dot{y}^2 \tag{59}$$

$$\ddot{\lambda}_1 u = -2x(y^2 - 1)\dot{x}\dot{\lambda}_1 - 2y(y^2 - 1)\dot{x}\dot{\lambda}_2 + 2x^2 y\dot{y}\dot{\lambda}_1 + 2xy^2\dot{y}\dot{\lambda}_2 \tag{60}$$

$$\ddot{\lambda}_2 u = -2x(x^2 - 1)\dot{y}\dot{\lambda}_1 - 2y(x^2 - 1)\dot{y}\dot{\lambda}_2 + 2x^2 y\dot{x}\dot{\lambda}_1 + 2xy^2\dot{x}\dot{\lambda}_2 \tag{61}$$

where $u = 1 - x^2 - y^2$. After some calculations we obtain a solution as

$$\begin{gathered} x(t) = cos(kt), y(t) = sin(kt), \\ \lambda_1(t) = \frac{1}{k} e^{-\frac{1}{2}\pi + kt} \cos(kt) C_4 - \frac{1}{k} \cos(kt) C_3 e^{(-1/2\pi - kt)} + C_1, \\ \lambda_2(t) = (C_2 k - C_3 e^{-\frac{1}{2}\pi - kt} \sin(kt) + \frac{C_4 e^{(-\frac{1}{2}\pi + kt)} \sin(kt)}{k}. \end{gathered} \tag{62}$$

The geometrical properties of the corresponding metrics are presented in Annex.

5 Conclusions

In this paper we illustrated the importance of the surface terms in finding some integrable geometries in three, four and five dimensions. The central role was played by the components of the angular momentum. We added a surface term $\dot{\lambda}_3 L_z$ to a free two-dimensional Lagrangian and we obtain a three dimensional geometry which is conformal flat and has a non-vanishing scalar curvature. We solved the geodesics equations and we found non-trivial solutions. We repeated the procedure and we investigated the case when two components of the angular momentum were added to a free three-dimensional Lagrangian.In this manner we obtain a five-dimensional metric admitting three Killing vectors. A singular

system was obtained if all components of the angular momentum were added. We treated this system within (HJ) formalism and we obtained three types of four dimensional metrics admitting three isometries. The procedure was extended to the curved space and the two-dimensional sphere was investigated in details.

6 Acknowledgments

One of the authors (D. B.) would like to thank T. L. Curtright for valuable comments and remarks. This work is partially supported by the Scientific and Technical Research Council of Turkey.

7 Annex

In the following the Christoffel components, Ricci scalar and the Weyl components of the obtained metrics are presented.

1. For the singular case we obtain three $5x5$ non-singular matrices from (36).Let us denote k as $k = x^2 + y^2 + z^2$.The first metric is given by (40).
The non-zero Christoffel components are given

$$\Gamma^1_{24} = -\Gamma^1_{15} = -\frac{z}{y}\Gamma^1_{34} = \frac{z}{x}\Gamma^1_{35} = \Gamma^1_{24} - \frac{x}{y}\Gamma^2_{15} = \frac{x}{y}\Gamma^2_{24} = -\frac{zx}{y^2}\Gamma^2_{34} = \frac{z}{y}\Gamma^2_{35}$$

$$= -\frac{x}{z}\Gamma^3_{15} = \frac{x}{z}\Gamma^3_{24} = -\frac{x}{y}\Gamma^3_{34} = \Gamma^3_{35} = -\frac{zx}{y}\Gamma^4_{15} = \frac{zx}{y}\Gamma^4_{24} = \frac{z^2x}{x^2+z^2}\Gamma^4_{34} = \frac{z^2}{y}\Gamma^4_{35}$$

$$= z\Gamma^5_{15} = -z\Gamma^5_{24} = \frac{z}{y}\Gamma^5_{34} = \frac{z^2x}{z^2+y^2}\Gamma^5_{35} = \frac{zx}{k}$$

Ricci scalar is given by $R = \frac{12}{x^2+y^2+z^2}$ and Weyl tensor components are

$$W_{1212} = -\frac{1}{3}\frac{2x^2+z^2+2y^2}{k^2}, W_{1213} = -\frac{1}{3}\frac{yz}{k^2}, W_{1214} = -\frac{1}{3}\frac{(z^2+y^2)z}{k^2}, W_{1423} = -\frac{1}{3}\frac{(z^2+y^2)x}{k^2}$$

$$W_{1215} = W_{1224} = \frac{1}{3}\frac{xyz}{k^2}, W_{1223} = \frac{1}{3}\frac{xz}{k^2}, W_{1225} = -\frac{1}{3}\frac{z(x^2+z^2)}{k^2}, W_{2335} = \frac{1}{3}\frac{xyz}{k^2}$$

$$W_{1234} = \frac{1}{3}\frac{z^2x}{k^2}, W_{1234} = \frac{1}{3}\frac{yz^2}{k^2}, W_{1313} = -\frac{1}{3}\frac{y^2+2x^2+2z^2}{k^2}, W_{2323} = -\frac{1}{3}\frac{2y^2+x^2+2z^2}{k^2}$$

$$W_{1523} = W_{2324} = \frac{1}{3}\frac{x^2y}{k^2}, W_{1314} = \frac{1}{3}\frac{(z^2+y^2)y}{k^2}, W_{1315} = W_{1324} = -\frac{1}{3}\frac{xy^2}{k^2}, W_{1323} = -\frac{1}{3}\frac{xy}{k^2}$$

$$W_{1325} = \frac{1}{3}\frac{(z^2+x^2)y}{k^2}, W_{1334} = -\frac{1}{3}\frac{xyz}{k^2}, W_{1335} = -\frac{1}{3}\frac{y^2z}{k^2}, W_{2334} = \frac{1}{3}\frac{x^2z}{k^2}, W_{2325} = -\frac{1}{3}\frac{(x^2+z^2)x}{k^2}.$$

The second metric is given by

$$b^{(2)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & -y \\ 0 & 1 & 0 & -z & x \\ 0 & 0 & 1 & y & 0 \\ 0 & -z & y & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix} \tag{63}$$

The non-zero Christoffel symbols are given by

$$\Gamma^1_{15} = \frac{y}{z}\Gamma^1_{24} = -\frac{y}{x}\Gamma^1_{25} = -\Gamma^1_{34} = \frac{x}{y}\Gamma^2_{15} = \frac{x}{z}\Gamma^2_{24} = -\Gamma^2_{25} = -\frac{x}{y}\Gamma^2_{34} = \frac{x}{z}\Gamma^3_{34}$$

$$\frac{x}{z}\Gamma^3_{15} = \frac{xy}{z^2}\Gamma^3_{24} = -\frac{y}{z}\Gamma^3_{25} = -\frac{xy}{z}\Gamma^4_{15} = \frac{xy^2}{x+y}\Gamma^4_{24} = \frac{y^2}{z}\Gamma^4_{25} = \frac{xy}{z}\Gamma^4_{34} = y\Gamma^5_{15}$$

$$\frac{y^2}{z}\Gamma^5_{24} = \frac{xy^2}{z^2+y^2}\Gamma^5_{25} = -y\Gamma^5_{35} == -\frac{x}{z}\Gamma^3_{34} = \frac{xy}{k}$$

$$Ricci \quad scalar \quad R = \frac{12}{x^2+y^2+z^2}.$$

The Weyl tensor components are

$$W_{1212} = -\frac{1}{3}\frac{z^2+2x^2+2y^2}{k^2}, W_{1213} = -\frac{1}{3}\frac{zy}{k^2}, W_{1214} = -\frac{1}{3}\frac{(z^2+y^2)z}{k^2}, W_{1215} = \frac{1}{3}\frac{z^2x}{k^2}$$

$$W_{1223} = \frac{1}{3}\frac{zx}{k^2}, W_{1224} = \frac{1}{3}\frac{xyz}{k^2}, W_{1225} = \frac{1}{3}\frac{z^2y}{k^2}, W_{1234} = \frac{1}{3}\frac{z^2x}{k^2}, W_{1235} = -\frac{1}{3}\frac{(x^2+y^2)z}{k^2}$$

$$W_{1313} = -\frac{1}{3}\frac{2z^2+2x^2+y^2}{k^2}, W_{1314} = \frac{1}{3}\frac{y(z^2+y^2)}{k^2}, W_{1315} = -\frac{1}{3}\frac{xyz}{k^2}, W_{1323} = -\frac{1}{3}\frac{xy}{k^2}$$

$$W_{1324} = -\frac{1}{3}\frac{y^2x}{k^2}, W_{1325} = -\frac{1}{3}\frac{y^2z}{k^2}, W_{1334} = -\frac{1}{3}\frac{xyz}{k^2}, W_{1335} = \frac{1}{3}\frac{(x^2+y^2)y}{k^2}$$

$$W_{1423} = -\frac{1}{3}\frac{(z^2+y^2)x}{k^2}, W_{1523} = \frac{1}{3}\frac{zx^2}{k^2}, W_{2323} = -\frac{1}{3}\frac{2z^2+2y^2+x^2}{k^2}, W_{2324} = -\frac{1}{3}\frac{x^2y}{k^2}$$

$$W_{2325} = \frac{1}{3}\frac{xyz}{k^2}, W_{2334} = \frac{1}{3}\frac{zx^2}{k^2}, W_{2335} = -\frac{1}{3}\frac{(x^2+y^2)x}{k^2}$$

$$b^{(3)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & z & -y \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & -x & 0 \\ z & 0 & -x & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix} \tag{64}$$

The non-zero Christoffel symbols of the second kind are given as

$$\Gamma^1_{14} = -\frac{z}{y}\Gamma^1_{15} = \frac{z}{x}\Gamma^1_{25} = -\frac{z}{x}\Gamma^1_{34} = \frac{x}{y}\Gamma^2_{14} = -\frac{zx}{y^2}\Gamma^2_{15} = \frac{z}{y}\Gamma^2_{25} = -\frac{z}{y}\Gamma^2_{34} = \frac{x}{z}\Gamma^3_{14}$$

$$-\frac{x}{y}\Gamma^3_{15} = \Gamma^3_{25} = -\Gamma^3_{34} = -\frac{zx}{x^2+y^2}\Gamma^4_{14} = -\frac{x}{y}\Gamma^4_{15} = x\Gamma^4_{25} = -x\Gamma^4_{34} = -\frac{x}{y}\Gamma^5_{14}$$

$$= -\frac{zx^2}{x^2+z^2}\Gamma^5_{15} = -\frac{zx}{y}\Gamma^5_{25} = \frac{zx}{y}\Gamma^5_{34} = -\frac{zx}{k}$$

Ricci scalar R=$\frac{12}{x^2+y^2+z^2}$, Weyl tensor components are

$$W_{1212} = -\frac{1}{3}\frac{z^2+2x^2+2y^2}{k^2}, W_{1213} = -\frac{1}{3}\frac{zy}{k^2}, W_{1214} = \frac{1}{3}\frac{zyx}{k^2}, W_{1215} = \frac{1}{3}\frac{z^2x}{k^2}$$

$$W_{1223} = \frac{1}{3}\frac{zx}{k^2}, W_{1224} = \frac{1}{3}\frac{(z^2+x^2)z}{k^2}, W_{1225} = W_{1234} = \frac{1}{3}\frac{z^2y}{k^2}, W_{1235} = -\frac{1}{3}\frac{(x^2+y^2)z}{k^2}$$

$$W_{1313} = -\frac{1}{3}\frac{2z^2+2x^2+y^2}{k^2}, W_{1314} = -\frac{1}{3}\frac{y^2x}{k^2}, W_{1315} = -\frac{1}{3}\frac{xyz}{k^2}, W_{1323} = -\frac{1}{3}\frac{xy}{k^2}$$

$$W_{1324} = \frac{1}{3}\frac{(x^2+z^2)y}{k^2}, W_{1325} = -\frac{1}{3}\frac{y^2z}{k^2}, W_{1334} = -\frac{1}{3}\frac{zy^2}{k^2}, W_{1335} = \frac{1}{3}\frac{(x^2+y^2)y}{k^2}$$

$$W_{1423} = \frac{1}{3}\frac{x^2y}{k^2}, W_{1523} = \frac{1}{3}\frac{zx^2}{k^2}, W_{2323} = -\frac{1}{3}\frac{2z^2+2y^2+x^2}{k^2}, W_{2324} = -\frac{1}{3}\frac{(x^2+z^2)x}{k^2}$$

$$W_{2325} = \frac{1}{3}\frac{xyz}{k^2}, W_{2334} = \frac{1}{3}\frac{xyz}{k^2}, W_{2335} = -\frac{1}{3}\frac{(x^2+y^2)x}{k^2}.$$

2. We are repeating the same procedure as above for the curved case. The metric given (55) has Ricci scalar is 0,$W_{1212} = -\frac{1}{u}$, Christoffel symbols of the second kind non-zero components

$$\Gamma^1_{11} = -\Gamma^3_{13} = \frac{x(1-y^2)}{u}, \quad \Gamma^2_{22} = -\Gamma^4_{24} = \frac{y(1-x^2)}{u}, \quad \Gamma^1_{22} = -\Gamma^4_{23} = \frac{x(1-x^2)}{u}$$

$$\Gamma^4_{14} = \Gamma^3_{24} = -\Gamma^2_{12} = -\frac{xy^2}{u}, \quad -\Gamma^1_{12} = \Gamma^3_{23} = \Gamma^4_{13} = -\frac{x^2y}{u}, \Gamma^2_{11} = -\Gamma^3_{14} = \frac{y(1-y^2)}{u},$$

$u = 1 - x^2 - y^2$.

$$g^{(2)}_{\mu\nu} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\sqrt{u}-\frac{y^2}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\sqrt{u}-\frac{y^2}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix} \tag{65}$$

Ricci scalar is R=0,$W_{1212} = -\frac{1}{u}$

The Christoffel symbols are

$$\Gamma^1_{11} = \frac{x(1-y^2)}{u}, \Gamma^1_{12} = \frac{x^2y}{u}, \Gamma^1_{22} = \frac{x(1-x^2)}{u}, \Gamma^2_{11} = \frac{y(1-y^2)}{u}, \Gamma^2_{12} = \frac{xy^2}{u}, \Gamma^2_{22} = \frac{y(1-x^2)}{u}$$

$$\Gamma^3_{13} = -x, \Gamma^3_{14} = -\sqrt{u}, \Gamma^3_{23} = -\frac{x^2}{y}, \Gamma^3_{24} = \frac{x}{y}\sqrt{u}, \Gamma^4_{13} = -\frac{x^2}{\sqrt{u}}, \Gamma^4_{14} = x$$

$$\Gamma^4_{23} = \frac{x(-1+x^2)}{\sqrt{-u}}, \Gamma^4_{24} = \frac{x^2-1}{y}$$

$$g^{(3)}_{\mu\nu} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & \sqrt{u}+\frac{x^2}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & \frac{xy}{\sqrt{u}} & x \\ \sqrt{u}+\frac{x^2}{\sqrt{u}} & \frac{xy}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix} \tag{66}$$

Ricci scalar is R=0, $W_{1212} = -\frac{1}{u}$. The Christoffel symbols are

$$\Gamma^1_{11} = \frac{x(1-y^2)}{u}, \Gamma^1_{12} = \frac{x^2 y}{u}, \Gamma^1_{22} = \frac{x(1-x^2)}{u}, \Gamma^2_{11} = \frac{y(1-y^2)}{u}, \Gamma^2_{12} = \frac{xy^2}{u}$$

$$\Gamma^2_{22} = \frac{y(1-x^2)}{u}, \Gamma^3_{13} = \frac{y^2}{x}, \Gamma^3_{14} = \frac{y}{x}\sqrt{u}, \Gamma^3_{23} = -y, \Gamma^3_{24} = -\sqrt{u}$$

$$\Gamma^4_{13} = \frac{(-y^2+1)y}{x\sqrt{-u}}, \Gamma^4_{14} = \frac{1-y^2}{x}, \Gamma^4_{23} = \frac{y^2}{\sqrt{u}}, \Gamma^4_{24} = y$$

References

[1] Dirac P. A. M., *Lectures on Quantum Mechanics* (Yeshiva University, New York, N.Y., 1964).

[2] *Carathéodory* C. *Calculus of Variations and Partial Differential Equations of First Order*, Part II (Holden-Day) 1967, Kastrup H. A. *Phys. Rep.* **101 no. 1-2**, (1983) 1.

[3] Güler Y., *Nuovo Cimento* **100 B**(1987) 251,
Güler Y., *J. Math. Phys.* **30**, (1989) 785.

[4] Güler Y., *Nuovo Cimento* **107 B**, *(1992) 1398.*

[5] Baleanu D. and Güler Y., J. Phys. A: Math. Gen. **34**, *(2001) 73.*

[6] Baleanu D. and Güler Y., Mod. Phys. Lett. B, **vol 16, no.13**,*(2001) 873.*

[7] Baleanu D. and Güler Y., Int. Journ. Mod. Phys. A **vol 16, no. 13** , *(2001) 2391.*

[8] Henneaux M.,Teitelboim C. and Vergara J. D., Nucl. Phys., **B387**, *(1991) 391.*

[9] Montesinos M., Vergara J. D., Gen. Rel. Grav **33** *(2001) 921.*

[10] Sheikh-Jabbari M.M. and Shirzad A., Eur.Phys.J. **C19**, *(2001) 383.*

[11] *Delfrate D. G.,Devecchi F. P., Marchioro D. F., Europhys. Lett.* **61**, *(2003) 15.*

[12] *Hanson A.,Regge T. and Teitelboim C.,Constrained Hamiltonian systems (Academia Nationale dei Lincei, Rome,1976),*
Sundermeyer K., Constrained Dynamics, Lecture Notes in Physics ,**vol.169** *(Springer-Verlag,New-York,1982),*
Henneaux M., Phys. Rep. **126**, *(1985)1,*
Henneaux M. and Teitelboim C., Quantization of Gauge Systems, (Princeton University Press, 1992),
*Faddeev L. D. and Jackiw R., Phys. Rev. Lett.***60**, *(1988) 1692,*
Govaerts J., Hamiltonian Quantisation and Constrained Dynamics (Leuven University Press, Leuven, 1991),
Shabanov S. Phys. Rep. **326**,*(2000) 1.*

[13] *Baleanu D. and Güler Y., Nuovo Cimento* **vol , no. 13** , *(2002) 123.*

[14] *Kalnins E. G., Kress J. M. and Winternitz P., J Math. Phys.* **43 (2)**, *(2002) 970.*

[15] *Curtright T.L. and Zachos C. K., New J. Phys.* **4**, *(2002) 83.*